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Math in Economics Series

Calculating Derivatives

This tutorial covers:

- what a derivative is
- how to calculate derivatives of linear, polynomial, natural logarithm, and exponential functions
- rules for computing derivatives: power rule, constant rule, sum rule, product rule and chain rule
- shows how to calculate second derivatives and partial derivatives

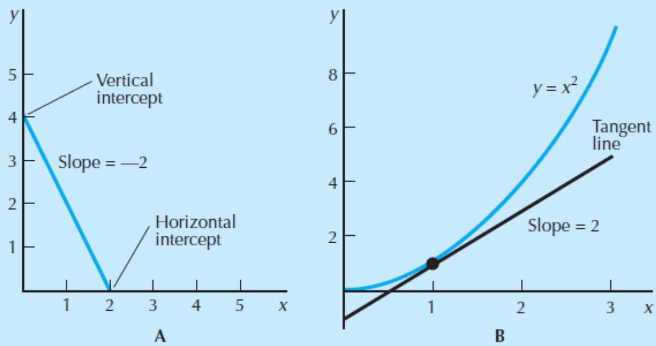
1 What Is a Derivative?

A derivative measures rate of change along a function.

Let's start with a function $f(x)$. To express this as an equation, we write $y = f(x)$. If the function is the line $f(x) = a + bx$, we all know that the rate of change along the function (i.e., the slope) is the constant b. More generally, the rate of change along the function $f(x)$ is $\frac{\Delta y}{\Delta x}$, the "rise" Δy over the "run" Δx, where the Greek letter Δ denote the change in the variable.

If we work with very tiny values of Δx, we're essentially expressing as a rate of change how $f(x)$ varies in the neighborhood of (or a tiny interval around) x. In fact, it's useful to think of a line that's tangent to the function at x. The **derivative** of $f(x)$ is the slope of the tangent line.

The left panel of Figure A.2 diplays a line. The derivative of that linear function is -2 at all values of x. In the right panel, the line that's tangent to the function $f(x) = x^2$ at $x = 1$ has slope equal to 2. The derivative of that function is 2 at $x = 1$. We have two ways to denote that derivative: $\frac{dy}{dx}(1) = 2$, and $f'(1) = 2$.



Slopes and intercepts. Panel A depicts the function $y = -2x + 4$, and panel B depicts the function $y = x^2$. **Figure A.2**

2 Rules of Differentiation

For each rule, we define a function f on the variable x; that is, we have $f(x)$, and $f'(x)$ denotes the first derivative. Other functions are $g(x)$ and $h(x)$, and a and b are constants.

Scalar Rule. If $f(x) = ax$, then $f'(x) = a$.

Sum Rule. If $f(x) = ag(x) + bh(x)$, then $f'(x) = ag'(x) + bh'(x)$.

Power Rule. If $f(x) = x^a$, then $f'(x) = ax^{a-1}$.

Example. If $C = 100 + 2q + 3q^2$, then $f'(q) = 2 + 6q$.

Product Rule. If $f(x) = g(x)h(x)$, then $f'(x) = g'(x)h(x) + g(x)h'(x)$.

Example. If $f(x) = (2x + 1)(3x + 2)$, then $f'(x) = 2(3x + 2) + (2x + 1)3 = 12x + 7$.

Reciprocal Rule. If $f(x) = \frac{a}{h(x)}$, then $f'(x) = -\frac{ah'(x)}{h(x)^2}$.

Example. If $f(x) = \frac{-15}{x^2}$, then $f'(x) = -\frac{30x}{x^4} = -\frac{30}{x^3}$.

Quotient Rule. If $f(x) = \frac{g(x)}{h(x)}$, then $f'(x) = \frac{g'(x)h(x) - g(x)h'(x)}{h(x)^2}$.

Chain Rule. For a function $h(y)$ with $y = g(x)$, we have $f(x) = h(g(x))$ and $f'(x) = h'(g(x))g'(x)$; more succinctly, $f' = h'g'$, or $f' = \frac{dh}{dy}\frac{dy}{dx}$.

Example. A firm's revenue $R(q)$ is a function of quantity produced and sold q, and the quantity produced is a function of labor input L: $q = F(L)$. Then $\frac{dR}{dL} = \frac{dR}{dq}\frac{dq}{dL} = R'(q)F'(L)$.

Practice Question

Derive the marginal cost function for the total cost curve represented by a quadratic cost function $C(y) = 6y^2 + 12$. To put it another way, if the total cost function is $C(y) = 6y^2 + 12$, what is the marginal cost function?

Logarithms. If $f(x) = \log(g(x))$, then $f'(x) = \frac{g'(x)}{g(x)}$.

This rule is an application of the chain rule with $h(y) = \log(y)$ and $y = g(x)$. This statement of the logarithm rule starts with $h'(y) = \frac{1}{y}$ and then applies the chain rule.

Example. If $f(x) = \log(5 + 2x)$, then $f'(x) = \frac{2}{5+2x}$.

Exponentials. If $f(x) = e^{g(x)}$, then $f'(x) = e^{g(x)}g'(x)$.

This is also an application of the chain rule, here with $h(y) = e^y$ and $y = g(x)$. In this case, $h'(y) = e^y$, and the chain rule delivers the result.

Example. If $f(x) = 667e^{.03x-3}$, then $f'(x) = 667e^{.03x-3} \times .03 = 20e^{.03x-3}$.

Constant Bases. If $f(x) = a^{g(x)}$, then $f'(x) = a^{g(x)} \log a\, g'(x)$.

Notice that exponentials are a special case with the constant a equal to e.

3 Marginal Revenue with Linear Demand

Suppose a monopoly seller faces a linear market demand curve. Express the linear demand curve as $p = a - bq$. Revenue is price times quantity, so $R = p \times q = (a - bq) \times q = aq - bq^2$ in this case. Since demand is linear, revenue is quadratic.

Marginal revenue is the derivative of the revenue function $R(q)$. Since $R = aq - bq^2$, the marginal revenue function is

$$R'(q) = a - 2bq$$

So the demand curve and the marginal revenue curve share the intercept a, and the marginal revenue curve is twice as steep as the demand curve.

4 Second Derivatives

The derivatives in the above examples are *first* derivatives because we take the derivative of the function only once. The **second derivative** is the derivative of the derivative of a function. If $f(x) = 2x^2 + 4x + 2$, the first derivative is $f'(x) = 4x + 4$. The second derivative of $f(x)$ is the first derivative of $f'(x)$.

$$f''(x) = \frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{df(x)}{dx}\right) = \frac{d}{dx}(4x+4) = 4.$$

(Either f'' or $\frac{d^2y}{dx^2}$ denotes the second derivative of $y = f(x)$.) The second derivative is a constant because the first derivative is a line.

Practice Question

If total cost function is $C(y) = 6y^2 + 12$, what is the slope of the marginal cost function?

5 Partial Derivatives

Partial derivative extends the concept of the derivative to functions of more than one variable. Suppose y is a function of two variables, x_1 and x_2. In this context, a derivative still refers to the slope of a tangent line, but now we must be careful about the direction of the change. A **partial derivative** changes one variable while holding the other variable(s) constant. In fact, the partial derivative of $f(x_1, x_2))$ *with respect to* x_1 is the derivative of the function holding x_2 constant. To specify the direction of the derivative, the language surrounding a partial derivative always includes the phrase *with respect to*.

With $y = f(x_1, x_2)$, we denote the partial derivative of y with respect to x_1 as $\partial y/\partial x_1$ or f_1.

For instance, if $f(x_1, x_2) = 3x_1^2 + 2x_2 + 6$, then the two partial derivatives are

$$\frac{\partial y}{\partial x_1} = 6x_1$$
$$\frac{\partial y}{\partial x_2} = 2$$

All our rules of differentiation apply to partial derivatives.

Marginal Rate of Substitution. A consumer's marginal rate of substitution is a function of his or her consumption of the two goods, x_1 and x_2. We represent the consumer's preferences by the utility function $u(x_1, x_2)$. The marginal rate of substitution function is the ratio of the two partial derivatives.

$$MRS = \frac{\frac{\partial u}{\partial x_1}}{\frac{\partial u}{\partial x_2}}$$

If the utility function is $u(x_1, x_2) = 2\ln x_1 + 3\ln x_2$, then the partial derivatives are

$$\frac{\partial u}{\partial x_1} = \frac{2}{x_1}$$
$$\frac{\partial u}{\partial x_2} = \frac{3}{x_2}$$

and the marginal rate of substitution is

$$MRS = \frac{\frac{2}{x_1}}{\frac{3}{x_2}} = \frac{2x_2}{3x_1}$$

Practice Question

Calculate the *MRS* function if the utility function is $u(x_1, x_2) = x_1^2 x_2^3$.

Additional Resources

For questions about this topic, see a **Calculus** tutor at the Dolciani Mathematics Learning Center (Hunter East, 7th floor) or any tutor in the Economics Tutoring Center (Hunter West, 15th floor).

The Dolciani Mathematics Learning Center also provides related tutorials on several platforms—CDs, DVDs, and online. Online access is through PLATO. Visit the front desk at the Math Learning Center to create a PLATO account.

Resources at the Dolciani Mathematics Learning Center

Topic	*Text DVDs*	*Generic DVDs*	*Tutorial CDs*	PLATO
derivatives/rates of change	U-1	F1-1, T3	C2.2	available
derivative as a function	U-1	F1-1, T3	C2.3	available
chain rule	U-1	F1-1, T4	C3.1	available

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